

CANDY-PASSING GAMES ON GENERAL GRAPHS, I

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We let G be an undirected graph and denote the vertex and edge sets of G by $V(G)$ and $E(G)$, respectively. The *candy-passing game on G* is defined by the following rules:

- At the beginning of the game, $c > 0$ candies are distributed among $|V(G)|$ students, each of whom is seated at some distinct vertex $v \in V(G)$.
- A whistle is sounded at a regular interval.
- Each time the whistle is sounded, every student who is able to do so passes one candy to each of his neighbors. (If at the beginning of this step a student holds fewer candies than he has neighbors, he does nothing.)

The candy-passing game was first introduced by Tanton [3], who defined the game for cyclic G . Tanton and Wagon proved that if G is an n -cycle then any candy-passing game on G with fewer than n candies terminates (see [2]). The first author [1] also studied the end behavior of candy-passing games on such G , showing that if the number of candies c is at least $3n - 2$, then the configuration of candies eventually stabilizes.

Here, we undertake the first study of the candy-passing game on arbitrary connected graphs G . We obtain a general stabilization result which encompasses the first author's [1] results for $c \geq 3n$.

Preliminaries. We call the interval between soundings of the whistle a *round* of candy-passing. Dropping the student metaphor, we will treat the candy piles as belonging to the vertices of the graph G . If, after some round, the amount of candy held by a given vertex will remain constant throughout all future rounds of the candy-passing game, that vertex is said to have *stabilized*.

We denote the degree of vertex $v \in V(G)$ by $\deg(v)$. Clearly, if some vertex $v \in V(G)$ has $k \geq \deg(v)$ candies at the beginning of a round, that vertex cannot end the round with more than k candies. Indeed, such a vertex will pass $\deg(v)$ pieces of candy to its neighbors and can, at most, receive one piece of candy from each of its $\deg(v)$ neighbors.

Finally, we say that a vertex $v \in V(G)$ is *abundant* if it holds at least $2\deg(v)$ pieces of candy. This definition implies:

Lemma 1. *After a finite number of rounds of the candy-passing game on G , the set of abundant vertices of G is fixed and each abundant vertex has stabilized.*

Proof. The total amount of candy on abundant vertices is nonincreasing. Furthermore, whenever an abundant vertex loses candy, that total decreases. Since the total amount of candy on abundant vertices cannot fall below zero, the amount

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of candy that can be lost by abundant vertices must be finite, so that the set of abundant vertices and the amount of candy on each such vertex must eventually become fixed. \square

Main Result. We may now prove our stabilization theorem:

Theorem 2. *Let G be connected. In any candy-passing game on G with*

$$c \geq 4|E(G)| - |V(G)|$$

candies, every vertex $v \in V(G)$ will eventually stabilize.

Proof. As a consequence of Lemma 1, we may assume that all candy that will be lost by abundant vertices over the course of the game has been lost, as this must happen within finitely many rounds. If there are no abundant vertices at this point, then the condition $c \geq 4|E(G)| - |V(G)|$ implies $c = 4|E(G)| - |V(G)|$ and that every vertex $v \in V(G)$ has $2\deg(v) - 1$ candies. In this case, all the vertices of G have stabilized.

We now assume that at least one abundant vertex remains. (Unless we are in the situation addressed in the prior paragraph, this is guaranteed by the condition $c \geq 4|E(G)| - |V(G)| = \sum_{v \in V(G)} (2\deg(v) - 1)$.) This vertex has stabilized, and so it must be receiving candy from all of its neighbors every round. Each of its neighbors v , therefore, must hold at least $\deg(v)$ pieces of candy every round. It follows that these neighbors must eventually stabilize, since these vertices pass candy every round and no such vertex may end a round with more candy than it began with. By an identical argument, the neighbors of these vertices must also pass candy every round, and so they, too, must eventually stabilize. As G is connected, continuing this argument shows that all the vertices of G must eventually stabilize. \square

Remarks. When G is an n -cycle, the condition $c \geq 4|E(G)| - |V(G)|$ is equivalent to the condition $c \geq 3n$. Our Theorem 2 generalizes the results of [1] for n -cycles with at least $3n$ candies. More generally, if G is connected and k -regular then the condition of Theorem 2 simplifies to $c \geq (2k - 1)|V(G)|$.

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